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THE FUNCTION NOTION IN ELEMENTARY MATHEMATICS

BY

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A. B. University of Illinois, 1913

THESIS

Submitted in Partial Fulfillment of the Requirements for the

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I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Henry Charles Zeis

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BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Master of Arts

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INTRODUCTION

The purpose of this paper is to present the notion of functionality as a means of vitalizing mathematics in the secondary schools. Attention is called to the nature of the criticisms that are at the present time being directed against high school mathematics. The contention that a proper emphasis upon functionality would make our mathematics more useful and real is then supported by evidence of the wide application of the function idea in numerous fields. The extent to which the function notion is treated in the mathematical courses of some foreign countries is compared with its use in our own schools. Finally, an effort is made to offer detailed suggestions on the treatment of this important idea in secondary mathematics.

CHAPTER I.

THE NEED OF A VITALIZING ELEMENT IN ELEMENTARY MATHEMATICS

Various evidences point to the fact that there exists considerable dissatisfaction with the mathematics as it is now taught in our high schools. Ideas antagonistic to mathematics, particularly algebra, have found expression at several of the recent meetings of the National Educational Association. "God bless the girl who refuses to study algebra; it is a study that has caused many a girl to lose her soul," is attributed to Superintendent Francis[1] of Los Angeles at the St. Paul meeting of the National Educational Association in 1914. And this radical exclamation evoked favorable comment in various newspapers discussing the incident.[2]

At the meeting of the Department of Superintendence of the National Educational Association in Cincinnati last February[3] the question "Can algebra and geometry be reorganized so as to justify their retention for high school pupils not likely to enter technical schools?" was a topic under discussion. This discussion is said to have taken an iconoclastic trend and received enthusiastic applause.

Commissioner Snedden[4] of Massachusetts contended that the burden of proof of the educational values of algebra and geometry in the

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1. School Review, October 1914
2. Ibid.
3. American Mathematical Monthly, March 1915, p. 88
4. Ibid., April 1915, p. 119

secondary schools rests with those who desire to have them continued as prescribed studies. He pointed out that the "disciplinary aim" has been seriously shaken by recent investigations and that the idea is now held by many that "mental discipline must be a by-product of any and all studies, pursued because of their ultimate worth in fitting the student for life in its cultural, civic, and vocational aspects." He admitted that the so-called "cultural" aims of mathematics are important, but argued that it is questionable whether they are realized by present methods of teaching. The speaker then suggested a thorough analytical consideration of mathematics from the standpoint, first, of its instrumental, or vocational value, and, second, its use as an agency in cultural education. As a result there should be offered, probably as electives, in both high schools and colleges, courses with these two objects, respectively, as the central aim of the study.

H. C. Morrison, Superintendent of Public Instruction of New Hampshire, in the Thirteenth Yearbook of the National Society for Study of Education, in an article entitled "Reconstructed Mathematics in High Schools," enters into a destructive criticism of mathematics as taught in high schools. "At present there is nothing to which most processes in algebra and geometry can be applied," he says, "except to more algebra and geometry. What is wanted is a kind of mathematics that will function in some other field than mathematics. Unless we can find some justification of our courses in the high school other than 'discipline,' many of them will travel the road of Greek. In our high schools, detailed, concrete aims, related to social needs, must replace formal aims." Mr. Morrison continues by suggesting different courses for students who

expect to follow different callings in life. The central idea in these courses is to include those things which are immediately useful. Girls engaged in acquiring the foundation for their normal life-work would be taught but little mathematics beyond the arithmetic of the elementary grades. Utility is the test that mathematics must satisfy in order to earn a place in the curriculum.

Further evidence of the attack upon mathematics is seen in the fact that colleges are inclined to lessen the mathematics prescribed for entrance. The University of Illinois recently reduced the amount of algebra required for entrance to any college, except engineering, from one and one-half to one unit.[1] The University of Chicago does not prescribe any mathematics for entrance.[2]

The American Mathematical Society,[3] recognizing the fact that influences were operating against mathematics, appointed a committee last year to consider the question "whether action is desirable on the part of the society in the matter of the movement against mathematics in the schools." The New England Association of Mathematics Teachers likewise has a committee, headed by Professor Tyler of Massachusetts Institute of Technology "to investigate current criticisms of high school mathematics."

It is evident that the critics of mathematics are too serious to be "laughed out of court." On the contrary, it behooves the friends of the subject to study their criticisms and, if possible, eliminate such features as may be objectionable. Teachers should endeavor to

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1. University of Illinois Annual Register 1913-14, p. 89
   Ibid. 1914-15, p. 71
2. College Entrance Requirements, U.S. Bureau of Education, Bulletin No. 7, p. 12
3. American Mathematical Monthly, op. cit.

enrich the content of mathematical instruction by including and emphasizing matter that relates to real life and to the practical experiences of the pupils. Then, too, the values of the study of mathematics, both direct and indirect, should be given more emphasis by teachers. Whatever improvement can be made along these lines will aid in supplying a motive, both to the student for studying mathematics and to the administrator for giving it a place in the curriculum.

In the present paper we propose to show that emphasis upon functionality is a means of improving mathematics in the direction indicated above. We expect to show that the function notion will carry with it much that relates to real life and to the practical experiences of students. But in order to realize the contribution which emphasis upon the function idea will make to the value of mathematics, we shall, in the next chapter, analyze the notion and observe its application in fields other than mathematics as well as in mathematics itself.

CHAPTER II.

APPLICATIONS OF THE FUNCTION IDEA

The mathematical concept function has quite an interesting history. The idea, as it is commonly accepted today, is the result of a process of evolution, during which it became more and more general, and explicitly defined. The first use of the notion is made by Leibnitz in 1694.[1] He used the term for any expression associated with curves, such as the length of a tangent or co-ordinates of a point. In 1718[2] John Bernoulli defined a function as an expression made up, in any manner whatever, of a variable and constants. The present notation, of which f(x) is typical, was first used by Euler and Clairot in 1737.[3] The use by Fourier of trigonometric series embraced a broader notion of function and led Dirichlet[4] to formulate a more comprehensive definition which embodies the notion as it is understood in mathematics today. This definition can best be understood after first considering another mathematical concept, viz. the variable.

A very general definition of a variable is as follows: "A variable is a symbol which represents any one of a class of elements."[5] These elements may be numbers,[6] points, curves, areas, in fact anything that may be classified. Thus, the tempers of men[7] may even be consid-

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1. Kantor, History of Mathematics, Vol. III, p. 215
2. Kantor, Ibid., p. 457
3. Encyclopedie des Sciences Mathematiques, Vol. 2, part 1, p. 6
4. Ibid., Tome II., Vol. I., Fasc. 1, p. 13
5. Young, Fundamental Concepts of Mathematics, p. 195
6. Bliss, in Young's Monographs of Modern Mathematics, p. 265
7. Whitehead, Introduction to Mathematics, p. 144

ered as classifiable, and if a symbol x be defined as representing any one of the possible tempers, then x would, according to the above definition, be a variable.

A few illustrations will tend to make clear the meaning of the term variable. Consider the symbol x as representing, (a) the number of dollars on deposit at the First National Bank of Champaign, or (b) the area[1] P_1P_2DC of Figure 1. Likewise, the symbol y may be defined as representing the time at which we are considering (a) above, or, as representing the curve P_1P_2 in example (b). In each case the symbols x and y may represent any one of a large class of elements and are therefore variables.

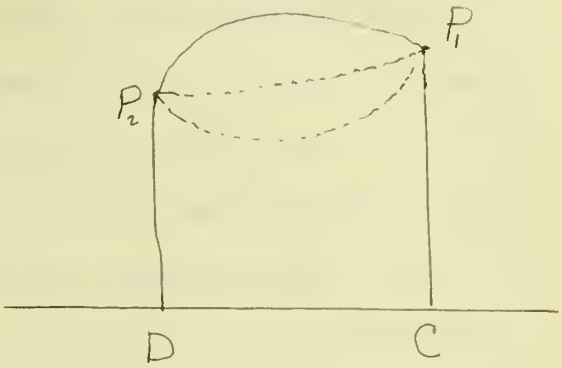

Fig. 1

It is evident that in the above illustrations there exists a correspondence between the elements of each of the two classes represented by the variable x, and the elements of the corresponding classes represented by y. Thus, if we fix the time, y, then the value of x, the number of dollars in the bank, is also fixed. Similarly, if the curve P_1P_2 is agreed upon, then the area P_1P_2DC is fixed. It is this correspondence between variables that forms the basis of the function concept. As formulated by Dirichlet the definition of a function may be stated as follows: "If two variables are so related that for each value that may be assigned to the one, there are determined one or more definite values of the other, the second variable is said to be a func-

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1. This illustration is given by Bliss, op. cit., p. 268

tion of the first."[1]

This definition does not imply that an algebraic relation exists between the variables. In fact, there are many functions whose algebraic expression is impossible, as, e. g. example (a) above. Hence, Bernoulli's definition was a very narrow one, and the notion of Leibnitz still more so. At present the function notion is applied to relations into which numerical considerations do not even enter, as "Reality is a function of the individual and the time,"[2] or "His temper is a function of his digestion."[3] The term function is here used in exactly the sense as defined above. Having the meaning of the function concept before us, we shall now proceed to examine into its applications, both in mathematics and in other fields.

The illustrations of functions given above show some of the ways in which this notion may enter into our daily activities. A great variety of examples like these might be cited. Let us first examine various phenomena of nature to see how the function concept enters into them.

The elements in nature all possess one characteristic property, namely variation or change. Temperature of the atmosphere, length of day, declination of the compass needle, tides, sunspots, weight of a body, elements of plant food in the soil, are all changeable. "Can you imagine one quantity in nature or in science not invented by man which is fixed, unchangeable, whose qualities and properties suffer no alteration?"[4] Not only do the elements of nature vary, but their variation

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1. Townsend and Goodenough, First Course in Calculus, p. 2
2. Carson, Mathematical Education, p. 45
3. Whitehead, op. cit., p. 144
4. Hedrick, School Science and Mathematics, Vol. 11, p. 54

depends on (is the function of) the variation of other elements of nature. Thus, the temperature of the atmosphere is a function of the season, direction of wind, proximity to sea, elevation, humidity, etc., all of which are variable elements. The declination of the compass needle varies with the place on the earth's surface, time of day, and also with the variation of sunspots.[1] The weight of a body, again, varies with its mass and its distance from the center of the earth. It will be seen, therefore, that in dealing with the relations between phenomena of nature, we deal with functions.

The science of physics deals primarily with natural phenomena. It is the purpose of the physicist to place the varying elements of nature under control[2] and establish relations between them, that is, set up natural laws. Any such law is very imperfect until it can be stated in exact mathematical form. But a physical law is simply a statement that, given a certain set of circumstances, certain events will follow, or stated in different language, the expression of one variable as a function of other variables. In his experiments, the physicist controls a certain number of elements and permits others to vary. In this way he can discover upon what elements and in what manner, a given property of a substance depends. Thus, Galileo, by dropping two balls of unequal mass from the leaning tower of Pisa and finding that they reached the earth in the same time, discovered that the distance through which a freely falling body falls is a function of the time of fall and not of its mass.

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1. Verbal Statement, Professor A. P. Carman, University of Illinois
2. Duff's College Physics, p. 5

The physicist also manipulates functions to find certain other related functions. For example, the time-derivative of the distance over which a body traversed equals the speed of the body, or the time-derivative of the speed gives the tangential acceleration.[1] Essentially, then, the physicist deals with functions.

In other fields of natural science the function, likewise, plays an important role. Let us examine some of the problems confronting the botanist. One of his important studies[2] has for its purpose the finding of relations such as those existing between growth and temperature, between light and gas formation of growing plants, or between growth and several variable elements, as light, temperature, and time. By means of experiments in which he controls the various elements entering, the botanist can obtain data exhibiting the functional relation between these elements. Although he may not be able to state these relations by means of formulae as the physicist does, he can generally express them graphically in a way which exhibits very plainly the salient properties of these functions, such as maxima, minima, rate of increase or decrease.

The bacteriologist is working with similar functional problems. As examples we may cite the following: The percent of decomposition of protein in raw milk by the natural enzyme of milk is a function of the time in which the enzyme acts.[3] Also the curve showing the ripening of cheese as a function of time may be plotted from experimental data[4] and the results will give valuable information for makers of cheese. Although

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1. Townsend and Goodenough, op. cit., pp. 75-6
2. Verbal statement, Professor C. F. Hottes, University of Illinois
3. Otto Rahn, Technical Bulletin No. 5, Michigan Ag. Col. Exp. Sta., p. 7
4. Rahn, cf. p. 16

the mathematical formulation of the relations discovered in bacteriology is not yet settled, it is claimed[1] that such mathematical expression is possible if all conditions are under control, which has heretofore not been possible due to inaccurate methods.

Scientific agriculture is to a considerable extent dealing with functional problems. An examination of the bulletins issued by the Illinois Agricultural Experiment Station as well as some of the bulletins issued by similar stations in other states and by the United States Department of Agriculture reveals the fact that the majority of these bulletins deal with some phases of agriculture which are fundamentally a functional problem. Following are some typical illustrations: (1) a table showing the number of stalks of corn per acre as a function of the distance between the hills; (2) for alfalfa-fed cows, the graph of the amount of milk produced per day for two weeks; (3) same graph for cows fed on timothy; (4) graphs on same axes showing the amount of milk produced by alfalfa fed and by bran-fed cows for two weeks; (5) a table showing the crop yields per acre corresponding to (a) varying amounts, and (b) varying kinds of fertilizers; (6) graph showing the increase in the temperature of a hog treated with cholera serum for hour intervals after the treatment. Examples such as these might be multiplied but the ones cited will suffice to show the extent and the nature of the application of the function concept in this important branch of science.

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1. Rahn, op. cit., cf. p. 5
(1) Illinois Agricultural Experiment Station Bulletin 26, p. 361
(2), (3), (4) Ibid., Bulletin 146, p. 136 ff.
(5) Ibid., Bulletin 99, p. 576
(6) Kansas Agricultural Experiment Station Bulletin 171, p. 177

The function notion also enters in the study of history. The Department of History of the University of Illinois uses outlines[1] in some of its courses which give directions for plotting such graphs as time-immigration and time-population. The object of this is to impress these relations upon the students, and the graph is considered the most effective way. The graph exhibits functional relations, and such properties as maxima and minima of these functions are of interest to the student of history.

Economics is another science in which functionality is of great importance. It deals with a great many problems which are studied through statistical data, real or theoretical. These data exhibit a functional relation, sometimes represented by graph, otherwise by tables. A typical problem of this kind is the relation between price and demand. Representing[2] the quantity of a commodity demanded by abscissae and its price by ordinates, this curve will take the general form shown in Figure 2. In the same manner the price-supply[3] curve may be constructed, appearing as in Figure 3. In these illustrations, both the supply and the demand are functions of the price, and conversely. Numerous other illustrations of functional relations

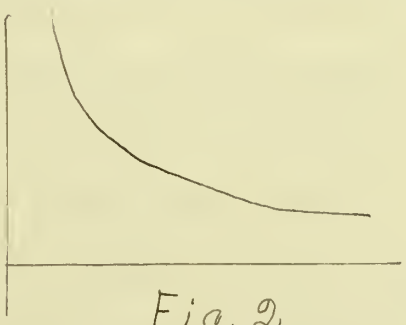

Fig. 2.

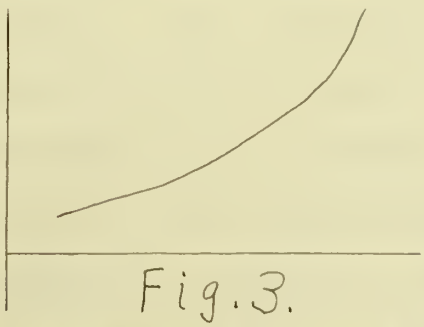

Fig. 3.

1. Thompson and Bogart, An Exercise Book in Economic History of the United States, Longmans Green and Company
2. Marshall, Economics of Industry, p. 65
3. Ibid, p. 198

are found graphically represented in standard texts on economics.[1] As another illustration we may cite the graph of the price of both raw and of manufactured articles as a function of the time.[2] This graph also exhibits the functional relation between these two sets of prices. The illustrations cited deal with fundamentals of economic theory and practice. They are, according to the definition of function, all function problems.

Numerous examples have been cited above, illustrating how the function idea finds expression in the contents of books and bulletins. Even less technical literature, such as popular magazines and newspapers contain many applications of this concept. Let us cite examples from the magazines in the Reading Room of the University of Illinois Library on March 21, 1915. Popular Mechanics, February 1915, page 273, contains a graphical representation of the relative amounts of artificial light used for the various times of the year, - a functional relation. Pure Products, February 1915, page 89, contains a diagram showing the number of microbes in a given volume of air, for air taken from various places, as country, city, office, factory, and schools; - again a functional problem. The Journal of Political Economy, February 1915, page 137, gives a table showing the relation between time in years for several years and (a) the number of railway employes, (b) tons carried one mile, and (c) passengers carried one mile. Monographs of Department of Health of the City of New York, September 1912, page 18, gives a table showing the number of bacteria in milk as a function of temperature and time. These illustrations show how extensively functional relations are met with

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1. Marshall, op. cit., pp. 96, 313
2. Seligman, Principles of Economics, p. 470

in the discussions of the live problems of the day in our current literature. In commenting upon this fact, Professor Felix Klein says, "We cannot open a newspaper any more out of which such a figure (graph) does not stare us in the face."[1]

Many of the illustrations cited as involving functionality deal with some of the every-day activities of the general public. In fact, the function notion underlies a great many phases of the business life. If a man goes on a business trip, the time required is a function of the distance and the mode of travel. The cost of the trip will be a function of the mode of travel, amount spent for other items, as lodging, amusement, and meals. The careful business man, before entering upon any business venture will study the variables of which the probable success of the venture is a function, such as probable competition, market conditions of raw and finished product, location, available labor, and so on. The probable success of the undertaking, again, is a function of the accuracy with which he has analyzed the situation and is able to control the variable elements of the scheme.

The ever-pressing problem of the high cost of living is nothing but a functional problem. What the variables upon which the cost of living depends are and the extent to which they enter to fix the function (cost of living) is the mooted question. When the price of sugar took a sudden leap in August 1915, President Wilson[2] ordered an investigation into the supply of sugar in the country. The same action was taken with reference to the price of flour. These steps were merely in accordance

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1. Klein und Schummack, Der Mathematische Unterricht an den Höheren Schulen, p. 34
2. Daily press

with the economic law that price is a function of supply and demand. In other words, the elements by which the price of commodities is determined must be analyzed and placed under control in order to control the cost of living.

To illustrate how even young children grasp functional relations in their experiences, Max Höfler in his "Didaktik des Mathematischen Unterrichts"[1] cites an instance of a nine-year-old boy. The boy had a fever, so his doctor requested him to take his temperature at stated intervals and record results. When the physician came on his next visit, Karl exhibited the results of his observations by means of a graph which he had drawn. The doctor showed surprise and inquired of the boy how he had come upon this idea. The boy replied, "Last year papa drew the curve showing baby's weight, so it occurred to me that I could show my temperature in the same way." A few years later when another baby came into their home, Karl was asked to record and plot its weight. Without being given any directions he did it, using the time as abscissa and weight as ordinate. One day he remarked to his father that there was no law governing the weight curve as was the case with curves in mathematics. The father tried to explain to him that there was, saying that the law involved was very complicated. Karl was not convinced, however, and replied: "There can surely be no mathematical law for the baby's weight, for when a new nurse came, the weight took a sudden jump." The notion of functionality had not been formally presented to the boy, but intuitively he saw a correspondence between the

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1. Page 23.

variable elements of his experiences. However, he was also shrewd enough to observe that this relation was not one that could be expressed by a mathematical formula.

Thus far we have considered applications of the function idea in other fields than mathematics. However, the notion is a dominant one in the science of mathematics itself. Trigonometry deals with periodic functions of angles. Analytic geometry represents functions graphically and sets up analytic expressions of given functional relations. Equations of curves are merely functional relations between the variable co-ordinates of any point on the curve. Calculus deals with differentiation and integration of functions. Generalizations of functional notation such as $F(x,y,z...)$ for any function of any number of variables materially simplify operations in higher mathematics, into nearly all phases of which functions enter.

The considerations in this chapter should be convincing of one thing, namely the fact that the notion of functionality has a very wide application in the activities of man. If, then, this concept enters prominently into the important fields of human endeavor, if the world's big problems are functional problems, then it would seem that functional thinking, and manipulation of functions, are far too important to neglect in the educative process of the child. Because this idea is so closely connected with the affairs of life, its introduction into secondary mathematics would greatly increase opportunities for relating this subject to real life. In other words, the function idea would vitalize algebra and geometry, make them real, and useful as well. It would supply the element of utility, the lack of which as we saw in Chapter I., has caused so much

criticism to be directed against our high school mathematics. Mathematics would then become, in part, a training in the habit of functional thinking, a habit that is, for every man or woman who possesses it, a most valuable asset.

CHAPTER III.

THE INTRODUCTION OF THE FUNCTION NOTION INTO THE SCHOOLS OF EUROPEAN COUNTRIES

The teaching of mathematics has for the last ten or fifteen years been characterized by a spirit of unrest in all progressive countries. It is the purpose of this chapter to trace, in so far as it involves the function notion, this agitation in Germany, Austria, and France.

Advanced ideas in the teaching of mathematics have probably been introduced most extensively in the states of the German Empire. In these countries there has been a distinct reform class among teachers of mathematics, advocating a definite reform program for the last twenty-five years. Sporadic efforts by individuals in the direction of these reforms date back almost to the middle of the nineteenth century. The substance of this reform movement is the early introduction of the notions of variable and functional dependence and the grouping of the courses in mathematics around these notions as a central aim.[1] It is with this signi-

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1. Die Entwickelung des Mathematischen Unterrichts-Reform in Deutschland. Band III., Heft. 1, p. 2.

Note: This work is one of the German reports of the International Commission on the Teaching of Mathematics. This report is the main source of the material in this chapter, and will be referred to in this paper as the "Report." A resume of this report is given in English by E. A. Price in Mathematical Gazette for December 1912 and January 1913.

ficance that the term "reform movement" is here used.

In order that the discussion of German schools which follows may be readily applied to our own schools, the following table, explanatory of the organization of the schools of Germany is given.[1]

TABLE I. NINE-CLASS SCHOOLS

Class	Abbreviation	Approximate Age
Oberprima	I.A	19.4
Unterprima	I.B	18.4
Obersecunda	II.A	17.4
Untersecunda	II.B	16.5
Obertertia	III.A	15.4
Untertertia	III.B	14.3
Quarta	IV.	13.2
Quinta	V.	11.8
Sexta	VI.	10.7

There are three kinds of nine-class schools, viz. the Gymnasium, the Real-Gymnasium, and the Oberrealschule. The distinction between them is primarily the emphasis placed on classical versus modern languages. Whenever the term "school" is used without qualification, it is this group of

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1. J. W. A. Young, Teaching of Mathematics in Prussia, p. 34

nine-class schools that is meant, as there is no essential difference in the mathematics taught in them. The above table permits a comparison of the ages of students in the German schools with those of our high school and college students. The mathematics of these German schools covers approximately the same ground as our courses to the close of the freshman year in college.[1]

Statements are sometimes made which tend to leave the impression that the movement for greater emphasis upon functionality is of comparatively recent origin,[2] but the fact of the matter is that the beginning of this movement dates back almost half a century. At first the efforts toward reform were concerned with the introduction of the function idea into the upper classes of the Gymnasium and similar schools, but of late years, the reformers would bring in functions as early as the Quinta or Quarta. The Prussian Course of Study of 1834 expressly forbade the teaching of the conic sections in the Gymnasium except for special classes of exceptionally able students.[3] In 1859, descriptive and analytic geometry and conics were permitted in the Real-Gymnasia of some of the provinces but most of the states followed the lead of Prussia in prohibiting the teaching of functions.[4]

Although the official courses of study of this time gave the reform ideas no encouragement or even expressly prohibited them, there were nevertheless teachers whose zeal for the reforms carried them beyond the prescribed regulations[5] and who taught such topics as conics, maxima,

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1. Young, Ibid., pp. 37-8
2. David Eugene Smith, Bulletin American Mathematical Society, February 1913, p. 251
3. Report, p. 4
4. Report, p. 4
5. Report, pp. 5, 7

and minima in the Oberprima, although they did not always explicitly emphasize the function notion.

At educational meetings these early pioneers frequently set forth their advanced ideas. Bertram, a teacher in a gymnasium at Berlin, in 1864, and Gallenkamp, also of Berlin, in 1873 pleaded for analytical geometry and even calculus in the gymnasium.[1] Similarly, in 1873, O. Simon argued that "analytical geometry, confined to the plane and presented by elementary methods, is indispensable to the Real-Schulen. Through it the function-idea is promoted."[2] Such proposals were, however, looked upon as revolutionary and received little attention at the time.

In 1877 there appeared a voice from an unexpected direction. E. DuBois-Raymond, a university professor, and a professor of physiology at that, severely criticized the way in which mathematics was taught in the schools. "Analytical geometry is still excluded from the Gymnasium," he declared, "which I think is a serious mistake. The representation of functions by curves or surfaces opens a world of ideas and teaches the use of one of the most productive methods by which the human mind has increased its power."[3] Even this declaration by a man whose eminence and disinterestedness commanded respect won few to the new cause. But one of those who did rally to its standard[4] was Hoffman, editor of the Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht, and for many years he proved himself the most ardent champion

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1. Report, p. 8
2. " p. 10
3. " p. 11
4. " p. 12

of the reform movement.

It was hoped that at the appearance of a revised Prussian Course of Study in 1882 the reform ideas would be given official recognition, but its advocates were again doomed to disappointment. Nevertheless, the reform ideas were slowly but surely gaining ground.

Among the most popular schoolbooks of this time were those by E. Bardey. In the eleventh edition of his Graduated Exercises,[1] issued in 1883, he added an appendix on graphic representation but without any directions as to how it should be used in connection with the remainder of the text. Instead of the term function, he used "dependent variable."

In 1887, Hoffmann, in his Zeitschrift, argues that no new text nor a new edition of existing texts should thereafter appear without a treatment of graphic work, so arranged as to make abstract calculation easier for the pupil.[2]

At a meeting of teachers of mathematics and natural science held at Jena in 1890,[3] a motion was brought forward to introduce the study of conics, both analytically and synthetically, into the gymnasium. However, the motion failed to carry. In the same year the Minister of Education at Berlin invited experts to offer suggestions for a new course of study.[4] The result was that in the Prussian syllabus of 1892 a provision was included, providing that the students of the upper classes be conducted into the especially important idea of co-ordinates,

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1. Report, p. 20
2. Zeitschrift, Ibid., 18 (1887) 589
3. Report, p. 24
4. " p. 26

and that some of the fundamental properties of conics be explained to them in the simplest possible manner.

In criticizing the above provision, M. Simon pointed out that if students were to be taught the "especially important" idea of co-ordinates, it would be necessary that they "acquire, at an early age, a grasp of the function idea, the most far-reaching and indispensable idea in algebra."[1] Although granted with reluctance, this provision in the syllabus offered an opening for the reform ideas.

Again, in 1897, at a meeting in Danzig of the Förderungsverein, a "Society for the Improvement of Teaching in Mathematics and Science,"[2] the proposal was made by A. Schülke that functions be introduced, beginning with the Untersekunda. Two years later, at a meeting of headmasters of the Rhine Provinces,[3] it was unanimously resolved that "the idea of co-ordinate, the function concept, and graphic representation of the course of functions shall be introduced in the Untersekunda in the Realschulen and into the Prima in the Gymnasium. These notions shall be abundantly applied in the teaching of mathematics and natural sciences." It was argued that the function concept should be explained and made concrete through co-ordinates as early as the Obersekunda, because the understanding of equations demands that the student enter more closely into the nature of algebraic functions.

The evidences considered thus far tend to show that the reform ideas had been slowly but surely coming to the fore and acquiring, near the end of the nineteenth century, an irresistible momentum. The pioneers

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1. Report, p. 27
2. " p. 28
3. " p. 29

in this movement were only isolated teachers and writers of text-books pleading for analytical geometry, but gradually their numbers were augmented by other teachers, university professors, and scientists. In the closing year of the century we saw unmistakeable evidence of the general dissemination of reform ideas, in the fact that a convention of headmasters unanimously passed a resolution calling for the introduction not only of analytical geometry into the Prima, but specifically mentioning the graph, the function, and co-ordinates in the work of the Obersecunda. Some of the advance guards as Simon, and as we shall see later, Höfler in Austria, go a step further and declare the function idea the natural aim of the whole mathematical teaching in secondary schools.

This agitation of almost half a century finally found more formal expression in the Prussian Lehrplan of 1901.[1] In outlining the work of the Oberprima the following statement appears: "At this stage there arises the opportunity for insuring a thorough comprehension of the idea of function with which the boys have already become acquainted earlier in the course." The implication here is that teachers continually refer to functions in elementary work, although not specifically prescribed in the course of study.

The twentieth century thus opened with official sanction of the new idea by the leading kingdom of the empire, and also a healthy, steadily growing support among teachers. It must not be understood, however, that the battle was already won, for a large majority of teachers were still indifferent towards the new movement and its general

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1. Report, p. 30

adoption was still a question that required earnest and constant agitation.

The first text-book that embodied the function idea in a systematic way in the Untersekunda appeared in 1902, but it was excluded from the schools until 1906.[1] However, the absence of emphasis upon functionality in the text is not a reliable test of what was taught, as German teachers are singularly independent of the texts used.

A memorable event in the history of the reform movement in Germany was the adoption of the so-called Meran Proposals in 1905. In these proposals the reform ideas are crystallized into tangible form and this document has ever served as a sort of Declaration of Rights of the Reform party.

The Meran Proposals originated in connection with a meeting of scientists in Breslau in 1904.[2] A committee of twelve was appointed to suggest reforms in the teaching of the sciences. Professor Klein of Göttingen, who, in an address before the meeting had advanced the claim that the function idea, graphically represented, should form the central notion of mathematical teaching, was a member of this committee. The report was laid before the meeting held in the following year at Meran, whence the name.

A copy of the Meran Proposals is given in "Klein und Schimack, Der Mathematische Unterricht an den Höheren Schulen," page 213. An English translation is found in the Mathematical Gazette, December 1911,

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1. Report, p. 35
2. " p. 38

pages 179 to 181. The proposals are in the form of a Lehrplan. The expression "Funktionales Denken" was coined at Meran to describe the fundamental thought in the new teaching. Although grammatically objectionable, the term has survived owing to its timely introduction.

The Meran Lehrplan is thoroughly permeated with the function notion. It is recommended that, as early as the Quarta, functionality be introduced into geometry in connection with the dependence of triangles upon given parts. In the Untertertia we find: "Continuation of exercises in the evaluation of formulae, continually insisting on the functional character of the values obtained by variations. Consideration of the effect produced on the whole figure by alterations in the length of the various parts."

In Obertertia: "Dependence of an expression upon one of its own variables. Graphic representation of simple linear functions and the use of this method in the solution of equations."

In Untersekunda: "Connection between coefficients and roots. Consideration of the variation of a quadratic expression in one variable due to changes in that variable, with graphical illustration. Graphical representation as a means of visualizing relations discovered empirically. Relation between length of side and size of angles in right triangles."

Obersekunda: "Graphic representation of logarithmic function. Solution of quadratic equations in two variables by calculation and by graphic representation. Trigonometry."

Unterprima: "Resume` of functions already dealt with. Regard for their rise and fall, with possible introduction of the ideas of differential and integral calculus."

Oberprima: "Conic sections. Review with generalizations."

There was some disagreement in the committee as to whether these proposals should recommend the introductinn of the calculus. They finally agreed upon the compromise that instruction in the Gymnasium should be carried to the threshold of the calculus.[1]

The leading spirit in the reform movement in Germany during the present century has been Professor Klein, who has already been mentioned in connection with the Meran Proposals. Klein is given credit by some with popularizing the function notion among the teachers of Germany and securing its general adoption.[2] The movement during the present century has been, in the main, to extend the function idea into the lower classes and to make this concept the central aim in mathematics.

In his work already referred to, which was published in 1907, Professor Klein declares an early emphasis upon "functional thinking" to be the great need of the time. "The early habit of functional thinking, alone will bring the student the full realization of his powers. By no means can it be a matter of an extensive 'theory' in these classes, but merely an emphasis on the natural dependence of given numbers upon each other. The student need not be scared by saying 'now we come to something entirely new, which really belongs to higher mathematics.' The idea of graphical representation of functions

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1. Report, p. 40
2. D. E. Smith Bulletin op cit.

can be taught through the simplest examples, such as y = 3x - 5."[1]

In this connection, Professor Klein offers an outline of a course of study in which, in Obertertia, "Algebra and geometry merge into the central idea: the function concept in graphical form." From the Obertertia on up, particular stress is laid each year on functionality. From the graphic representation of a function such properties as slope and tangents can be derived, even in the beginning, leading finally, in the Unterprima, to the notion of differentiation and integration.

Doctor Rudolph Schmack, the compiler of the German report to which repeated references have been made, gives a plan of a course of study[2] embodying reform principles. Doctor Schmack is the principal lieutenant of Klein in the reform movement. In this Lehrplan functions of first degree, treated analytically and graphically, are already introduced in the Untertertia. It is evident that as time passes, the tendency is to introduce the function idea earlier in the course. It will be observed from Table I. that the average age of pupils in the Untertertia is between fourteen and fifteen years, which is the average age of American children entering high school.

That the reform ideas continued to make steady progress in Germany is evidenced by the favorable comments upon the movement brought out by the discussions in the various meetings of scientists and teachers. At the annual meeting of scientists at Salzburg in 1909, E. Hoppe, a teacher in a Gymnasium at Hamburg, reported most favorably upon his

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1. Klein, p. 37 ff.
2. Report, p. 136

experiences with the use of the function idea in his school. "The students," he says, "have not only easily mastered the work but also with greater interest and less burdensome home-work, and the results of the examinations encourage a general adoption of the idea in the curriculum."[1]

At a meeting of scientists and teachers in Basel in 1907,[2] several members reported successful experiences with the early introduction of the reform ideas. Professor Wittig of Dresden favored preparation for the idea of functional dependence as early as the Sexta. Professor Grossman of Zürich reported that in his school, teachers are urged to bring out the function idea incidentally in connection with daily work. Professor Grimsehl of Hamburg expressed the opinion that we should not hesitate to introduce functions early, "for," he says, "life offers nothing but variable quantities and functional relations." "The geographies," he added, "make abundant use of graphical work as early as the Sexta."

In the schools of Saxony[3] and Bavaria the reform ideas of the Meran Proposals received a hearty welcome. In the Bavarian course of Study of 1907 the following provisions are found in connection with equations of the first degree: "Examination into the dependence of quantities upon one another, with a view to the gradual unfolding of the function idea." In every grade the functions connected with the subject-matter are particularly emphasized.

Responses to a questionaire in 1908[4] revealed the fact that

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1. Report, p. 49
2. Zeitschrift, Mathematischen und Naturwissenschaftlichen Unterricht 39 (1908) 80 ff.
3. Report, p. 72
4. " p. 74

among the schools of East and West Prussia, half of the complete schools reporting (13 out of 26) had already introduced the function concept. "All report that the results are very satisfactory and that the students fully comprehend the notion and take pleasure in being instructed in it."

Answers to a similar questionaire sent from the schools of Saxony[1] contained only one report in opposition to the teaching of the function concept. Most of them favored the earliest possible introduction of functional dependence.

The 1909 Lehrplan of Hesse likewise warmly endorses the new ideas. "Functional thinking, justly placed in the foreground, is to be begun in the Sexta and emphasized increasingly in the succeeding classes so as to become a habit with the students."

Even the Lehrpläne[2] for the Girls' Schools of Prussia show this progressive spirit. "In order to train the students in the habit of functional thinking, functional relations shall be emphasized at every opportunity in algebra and geometry."

Other evidence of the progress of the reform movement may be obtained from the text-books used in the schools, but we must again remark that in German schools the teachers are not slaves of the text-books.

The mathematical texts in use about the year 1910 may be divided into several groups.[3] One group consists mostly of books that are revisions of former texts, and they treat functionality for the lower classes only in an appendix, and not in the text proper. In the upper classes the function concept is given somewhat greater prominence, but nowhere is it made an organic part of the subject matter. At the begin-

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1. Report, p. 75
2. " p. 83
3. " p. 123

ning of trigonometry the authors of these books still find it necessary to give an exposition of functions. Their books seem to be revised so as to include functionsbecause a popular demand for them has arisen, rather than because the authors are in sympathy with the movement.

The second group[1] of texts is in complete accord with the reform idea that the function notion and graphic representation shall permeate instruction from the beginning. Frequently, in the algebra, the function notion controls the order of presentation of the subject matter. In the Schwab-Lesser texts the function idea is brought out in connection with the evaluation of formulas and graphic representation early in the course. The treatment of functions becomes more formal and rigorous in the succeeding years.

In the Behrendsen-Götting texts, the simple algebraic quantities, such as sum, product, and difference, are analyzed for their functional dependence upon their component parts. Graphic representation of empirical functions leads to equations of first degree in one and two variables. Also, in geometry, there are references to functionality in the Quarta in connection with the construction of triangles with three given parts.

Several other truly reform texts might be mentioned. Geometrie, by H. Koppe and J. Diekman; Algebra, by Heilerman and Diekman; Geometrie, by Walther, all breathe the same reform spirit as the two books reviewed above.

In order to give an idea of the detailed application of the function idea in these texts, the following quotations from Schwab: "Geom-

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1. Report, p. 125

etrie für die Mittleren Klassen der Realanstalten" are here given:

1. The vertical angle of an isosceles triangle is a function of the base angle.

2. One of two supplementary angles is a function of the other.

3. The sum of the angles of a polygon is a function of the number of sides.

4. The third angle of a triangle is a function of the sum of the other two.

5. An exterior angle of a triangle is a function of the interior angles.

6. In a right triangle, either acute angle is a function of the other.

7. The medial of a trapezoid is a function of the parallel sides.

8. In the same circle or congruent circles the arc, the chord, the sector, and the segment are functions of their central angles.

9. The length of a chord of a given circle is a function of its distance from the center.

10. Given angle C, side c, and R, the radius of the circumscribed circle, then any one element is a function of the other two. If C is acute and increases to 90°, R decreases to $\frac{c}{2}$, then increases.

11. The area of a square is a function of its side.

12. The area of a triangle is a function of its three sides.

13. In a regular inscribed polygon, a side, the radius of the inscribed circle, the perimeter, and the area of the polygon, may be expressed as a function of the circle.

14. The length of an arc is a function of the central angle and of the radius of the circle.

15. The area of a sector of a circle is a function of its central angle and radius.

16. The area of a circle is a function of its radius.

17. The diagonal of a cube is a function of its edge and conversely.

18. The volume of a cube is a function of its edge.

19. The volume of a prism with a given base is a function of the altitude; with a given altitude, a function of the base. With a given volume the height is a function of the area of the base, and conversely.

20. The volume of a right regular prism with a given radius of the circumscribed circle of the base and a given altitude, is a function of the number of sides. (What is the limit as $n = \infty$?)

The Behrendsen-Götting, Lehrbuch der Mathematik, Unterstufe B, für Realschulen, contains eleven similar references to functions.

We shall now briefly trace the reform ideas in Austria. It is a significant fact that since 1849 analytical geometry has had a place in the Austrian Lehrplan. By 1884 the reform ideas were accorded a prominent place in this Lehrplan.[1] With reference to to analytical geometry in the Unterprima (top class but one) these provisions are found: "The concepts of variable and constant, independent variable and function, cannot be avoided in a rigorous treatment of the subject. The mathematics already known to the pupil offers sufficient opportunities for relating these notions in order to make them understood. The beginning should be made with representation of linear equations. The general problem of the graphical representation of a variety of simple forms of equations should

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1. Report, p. 22

precede the full treatment of the theory of straight lines. To emphasize the general application of this method, graphs should be drawn, illustrating some of the laws of Physics, while empirical functions, e. g. curves of mortality or temperature, should not be omitted."

In Chapter II. reference was made to a book by Max Höfler, now a professor at the University of Vienna. In an article written in 1887[1] this author expresses the following views. "There is no better method of familiarizing a pupil with the function-idea than by leading him to continual plotting of graphs of all kinds. I maintain that the habit of considering the subject-matter of elementary algebra and the trigonometrical ratios expressly as examples of functions, should be the natural aim of the whole mathematical teaching in the secondary schools." Concerning his own experiences with graphical representation of functions he writes[2] that during his first year of teaching in 1876-7, in the Gymnasium at Vienna, he used graphical representation of the logarithmic and trigonometric functions, although considered an outsider by his associates and the school inspector on account of his advanced views.

Professor Höfler's book, previously referred to, "Didaktik des Mathematischen Unterrichts", is pronounced one of the best of its kind by Professor D. E. Smith.[3] In this book, the author outlines a plan for leading up to the function notion inductively with empirical functions graphically represented, without any reference to co-ordinates. He lays special emphasis on "funktionales Anschauen" and "funktionales Denken."

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1. Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht, 18 (1888) 25
2. Report, p. 23
3. Bulletin loc. cit.

The general notion of function is gradually to be built up as the applications broaden. The final generalization is to take place in the Prima.

In the Lehrplan for the Realschulen of Austria for 1909, the function idea is incorporated into mathematics from the lowest classes up.

In France, the function notion has won for itself a prominent place in mathematical teaching. In that country there arose, about the beginning of the present century, a demand by the engineers for better instruction in mathematics for those entering their profession.[1] One of the results of this agitation was the movement to introduce the function idea. Leading mathematicians, such as Tannery, Borel, Fehr, espoused the reform cause.

In L'Enseignement Mathematique (1905), pages 176 to 187, H. Fehr presents the claims of functionality for recognition on the basis of its manifold applications. He outlines an elaborate plan for the treatment of this notion in algebra, beginning with simple, linear functions.

But as early as 1902[2] the reform movement had gained a foothold in the French schools. In the French course of study of that year the following provisions are found for the classes corresponding to Obertertia: "Graphic representation of $ax + b$, of x^2, of $\frac{1}{x}$, etc., interwoven with the theory of equations. In the following year functions of the type $\frac{ax + b}{cx + d}$ are to be treated. All of this is to be done with the simplest numerical examples."

Further evidence that these ideas were not merely recommended in

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1. D. E. Smith, personal letter
2. Klein und Schimack, loc. cit., pages 41-2

the course of study but actually carried out, is found by their appearance in some of the texts in use at that time. A mathematical series by the great mathematician, E. Borel, incorporates these reform ideas in the body of the text.[1]

From the considerations in this chapter it is evident that the function concept is no longer an experiment in the schools of Europe. Not only are the text-books in harmony with this idea, not only do the leaders in the teaching of mathematics advocate it, but the teachers who do the actual instructing are thoroughly imbued with the reform ideas. The long struggle for the introduction of analytical geometry into the upper classes led to greater and greater emphasis upon functionality, until it is now in general use in the schools of Germany, Austria, and France. It is being taught very generally in the classes corresponding to those of our first-year high school, and throughout the courses is made the central aim.

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1. Klein und Schumack, loc. cit., p. 42

CHAPTER IV.

THE INTRODUCTION OF THE FUNCTION NOTION INTO THE SCHOOLS OF THE UNITED STATES

In our country the movement towards reform of mathematical teaching through the introduction of the function notion has not attained the impetus which it has acquired abroad. To determine the extent to which it actually has found favor in America we shall examine the pedagogical literature pertaining to this notion, and the mathematical texts used in the high schools in this country.

In striking contrast to the wealth of literature discussing the function idea in the German language there is a dearth of material by American writers, or in fact by any writers using the English language. A bibliography[1] compiled by David E. Smith and Charles Goldziher contains twenty-six articles on the function concept. One of these is written in French, two in Swedish, two in Bohemian, twenty-two in German, and none at all in English.

Professor E. H. Moore of Chicago University in his Presidential Address "On the Foundations of Mathematics," delivered before the American Mathematics Society, December 29, 1902,[2] takes a progressive stand on the teaching of elementary and secondary mathematics. In the main he contends for emphasis on the applied side of mathematics by means of a correlation of algebra, geometry, and physics, and teaching by the laboratory method.

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1. Bulletin No. 29, United States Bureau of Education
2. Science, Volume 17 (1903) 401-16

He would introduce graphical methods early, and study the graph as depicting phenomena depending upon each other. He suggests emphasizing the concrete meaning attached to the fact that, at a certain point, the curve is going upward, downward, or is horizontal. His emphasis on functionality is implied, however, rather than explicitly stated.

In an address entitled "Topics for Elementary Algebra," delivered in 1910 before the Kansas Association of Mathematics Teachers, Professor E. R. Hedrick of the University of Missouri takes a stand in favor of the function idea. "The chief direct value of algebra, in fact the real subject matter of algebra," he says, "consists in the study of variable quantities and the ability to control and to interpret the relations between them." As possibilities for the study of functionality the speaker suggested scientific formulae, proportion, variation, substitutions, logarithms, and graphic representation. "Graphic representation, more than any other topic, shows clearly the ways in which one quantity may vary owing to its dependence upon another."[1]

In his monograph "The Function Concept and the Calculus," Professor Bliss[2] of Chicago proposes the function concept as a unifying element in our mathematics. He suggests that each course in elementary mathematics should deal with a particular class of functions. This treatment of functions is to be as complete as is possible at the stage in which the student finds himself. It is a conspicuous fact, however, that this writer would begin to lay such stress upon functions only in the freshman year in college, rather than in the more elementary grades, as is done in the schools of European countries.

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1. School Science and Mathematics 11 (1911) 51-60
2. Monographs of Modern Mathematics by J. W. A. Young, p. 264

J. W. Young, in his "Fundamental Concepts of Mathematics," calls attention to the importance of the function notion thus: "The notion of functionality is one of such fundamental importance in daily life that it should be introduced as early as possible into our school curriculum. There seems to be no good reason why this conception should not be introduced in a first course in algebra or geometry, if not earlier. The pupil's experience can be drawn on for the most varied examples of this notion, and if judiciously used should contribute much toward arousing his interest in mathematical problems."[1]

In an address before the New England Association of Mathematics Teachers in 1912, Professor D. E. Smith[3] refers to the function idea as one about which much would be heard in this country within the next decade. He cautions against blindly adopting the notion without first making a thorough study of its purpose and significance.

Of the works on the teaching of mathematics, the following three were examined for their views on the function notion: "Teaching of Mathematics," by J. W. A. Young, "Teaching of Elementary Mathematics," by David Eugene Smith, and "Teaching of Mathematics in the Secondary Schools," by Schultze.

Professor Young[2] refers to the idea of functionality as a simple one in itself and a commonplace of the pupils' experience. He cites such illustrations as "the distance traversed by a train running at uniform speed is a function of the time." He says further, that even the beginner of algebra can understand this notion and represent it on squared paper.

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3. Ginn and Company, 1913
2. Page 308
1. P. 197.

No other suggestions are made as to the time or manner of treating the subject in the algebra course.

Professor Smith[1] urges the importance of functionality and cites common examples to illustrate the manner in which the concept enters into our daily experiences. He contends that this notion should be treated in the mathematics courses. However, he says, that the function notion is not necessarily foreign to the traditional way of presenting algebra through evaluation of formulas; but in the usual method of presentation, the function idea is not especially emphasized and the name made prominent.

The subject of graphs is discussed at considerable length by Professor Schultze.[2] One of the arguments in favor of teaching graphs is that through them the student "acquires a clear notion of one of the most important notions of advanced mathematics, viz. functionality." "It is not sufficient," he continued, "that the student construct graphs; they should also be interpreted. Thus, in a temperature graph, the student should find the temperature at a given time, and conversely, also maxima, minima, and the time of most rapid change." He does not emphasize functionality in itself, but incidentally as a by-product of the study of graphs.

Among the algebra texts used in our secondary schools there is one class of books which give only an incidental mention of the term function by defining it and citing a few illustrations. The graph is given some prominence, particularly in the solution of simultaneous equations and in the plotting of statistical data. In fact, the graph is used for this purpose by all the standard texts of today. However, in the first class of books

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1. Page 164
2. Pages 333 ff.

under consideration, the real significance of the graph and its properties, the representation of the relation between dependent and independent variable, is not brought out. The graph is usually constructed in a mechanical way as a representation of an equation in two variables. Among the texts of this class may be mentioned those by Hawkes, Louby, and Touton,[1] and by Slaught and Lennes.[2]

Professor Hedrick,[3] in his text, puts considerable emphasis upon the meaning of the graph, especially the graph of variable phenomena occurring in nature. A bare allusion is made to functions in the discussion of variation. Yet, functionality does not enter into the body of the subject-matter to the degree expected after reading Professor Hedrick's inspiring address referred to above.

A second class of texts devote several articles, or perhaps a chapter to functions, besides the usual treatment of the graph. Functions are usually defined, evaluated, and graphically represented. A serious objection to this sort of treatment lies in the fact that functions are introduced as an isolated part of the subject-matter. The learner, if he gets any idea about functions at all, will think of them as being a separate topic in the algebra, rather than a principle underlying, unifying, and vitalizing the entire subject-matter.

Among the books of the above class, we may mention the texts by Wentworth and Smith[4] and by Collins.[5] Collins introduces a set of drill exercises in the stating of functions. A fairly good discussion of the

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1. First Course in Algebra and Second Course in Algebra, Ginn and Company, 1911
2. First Principles of Algebra, Allyn and Bacon, 1912
3. Algebra for Secondary Schools, American Book Company, 1908
4. School Algebra, Ginn and Company, 1913
5. Practical Elementary Algebra, American Book Company, 1908

function concept with applications appears near the end of the book. If this discussion were given earlier and consistently carried through, the effort would deserve considerable credit. Wentworth and Smith state in the preface of their Algebra that they attempt to set forth the modern idea of function without carrying the work to an unwarranted extreme. They introduce the subject through evaluation of formulas anf of algebraic expressions. Equations and statistical data are plotted. The function, as such, is not named in Book I. In Book II. a chapter is devoted to functions, including a formal definition, evaluation, and plotting of functions. Such a treatment can hardly be said to "set forth the modern idea of functions," if we take as a standard the treatment of the subject in German texts. As mentioned before, the treatment of functions is too isolated; functions are not mentioned outside of the chapter dealing with them.

The books of a third class make an attempt to incorporate the notion of function and variable into the subject-matter somewhat more extensively. The best representative of this class is Book I. of the mathematical series by Long and Brenke.[1] Three chapters are here devoted to functionality and its applications. The notion is approached concretely through examples of variable phenomena of nature. The graph is used profusely as a means of representing functions. Algebraic and empirical functions are plotted. Plotting is led up to by representing the values of the function by means of perpendicular lines. On the whole, this effort is the nearest approach to a recognition of the importance of functionality to be found in any secondary text by an American author.

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1. Algebra, First Course, The Century Company, 1913

In their new high school algebra, Rietz and Crathorne[1] attempt to introduce the function notion early into the body of the text, although they try to keep their efforts in this direction under cover, and do not introduce the name function until the latter part of the book. Drill exercises are provided in evaluating expressions in one letter and studying the maximum and minimum values of such expressions. Scientific and historical data are plotted and equations on two variables solved by graphs. But for teachers not accustomed to emphasize functionality it would seem the texts should come out openly, present functions as such, and point out application of the idea throughout the course, lest the inexperienced teacher fail altogether to grasp the intentions of the authors. It is a pretty safe assumption that the average teacher will use this text throughout the year without considering the function notion as being entitled to any more than passing mention.

Although we are dealing primarily with secondary mathematics, attention is called to a recent effort toward functional emphasis in a course in mathematics for first-year college work. We refer to the work entitled "Elementary Mathematical Analysis," by Professor C. S. Slichter of the University of Wisconsin (McGraw, Hill and Company, 1914). In the preface the following statements are made: "The emphasis of the book is intended to be upon the notion of functionality. Illustrations from science are freely used to make this concept prominent. The student should learn early in his course that an important purpose of mathematics is to express and to interpret the laws of actual phenomena and not primarily to secure certain computed results.

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1. School Algebra, Henry Holt and Company, 1915

"The material of the book is grouped about the consideration of three classes of functions: the power function of which $y=ax^n$ is taken as a type; the simple periodic function, $y=a \sin mx$; the exponential function, $y=a^x$. The plan is to enlarge the elementary functions by the development of the fundamental transformations applicable to these and other functions."

It is not our purpose here to review this book from the standpoint of its merits as a text, but only in so far as it presents the function notion. At the outset the author leads up to the notion of a function by considering concrete examples of empirical functions. These are followed at once by drill in stating algebraic functions which express given relations. Mathematical functions are then plotted, beginning with the linear functional equations in two variables. Slope and intercept of the line are discussed. The power function is then introduced and its properties are discussed. Science is continually drawn upon for illustrations. Modifications of the various types are then brought about by means of transformations of axes. Transcendental functions are treated in a later chapter.

It must be granted that this book is consistent with the statements of its aim in the preface; functionality is the central aim. The material of the book is related to life, particularly to engineering, hence for the engineer it is no doubt practical. The use of graphical methods gives a sense of reality to the operations and results. Possibly graphical computations are carried to extremes. Too much time would be required for their execution. Theoretically, at least, the book is a faithful application of the reform doctrine. Its teachability will, however, have to be tested in the classroom.

From our consideration of the extent to which the function notion has been incorporated into mathematical teaching in the United States, we must conclude that we are far behind the European countries in this movement. The reform ideas have not been advocated as extensively by our leaders, nor have they found their way into our classrooms to the extent to which they have been introduced in Europe. There is no evidence that teachers are at all enthusiastic about the movement, and most writers of texts have presented the function notion in a half-hearted manner. Even such writers as Young, Hedrick, and Smith, who have expressed themselves as favoring the teaching of the function notion, do not consistently incorporate this notion into their texts. Except in so far as the function is connected with the graph, none of the American texts, with one exception perhaps, can be said even to approach the emphasis placed upon functionality by present-day German authors. Our consideration in Chapter II. of the wide application of functionality in life and in science, and in Chapter III. our consideration of the encouraging experiences with the teaching of this notion in European schools, make suggestions for the extension of this notion in our teaching of mathematics seem highly desirable.

CHAPTER V.

PEDAGOGICAL SUGGESTIONS

In this chapter we shall offer some suggestions that may be helpful in the teaching of functions in high school mathematics. Our plan is to treat functions, not as an isolated topic of algebra, but as a concept inherent in a large part of the subject-matter taught at present. For example, a graph represents a functional relation between two variables. It is this relation, which is commonly overlooked, that we would especially emphasize in the teaching of the graph. Likewise, an equation in two or more variables expresses a functional relation between these variables. These examples, and others that we shall discuss throughout the chapter, show that the function idea underlies a large part of the traditional subject-matter of algebra. In fact, algebra has been defined as "the calculus of functions."[1]

For carrying into execution the above plan of emphasizing functional relations in algebra, we offer the following suggestions.

1. In answer to possible objections to the teaching of the function notion in the high school, we contend that

(a) It is not proposed to teach "function theory" in the high school.

(b) The function notion is not too difficult for high school students.

(c) Whatever additional time may be required for special emphasis upon functions may be obtained by omitting other more abstract material.

2. Narrow definitions of function should be avoided.

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1. Comte, quoted by D. E. Smith Teaching of Elementary Mathematics, page 162

3. There are four ways in which functions may be represented, namely, by an analytic expression, by graph, by sets of tables, or by a statement in words.

4. The function notion should be introduced at the beginning of the algebra course through the graph. The concept need not be analysed or defined on first presentation.

5. The notions of variable and correspondence should be deduced from discussions of the graph.

6. Drill in functional thinking may be obtained by plotting a large number of sets of familiar data.

7. Formulae from arithmetic should be evaluated and plotted whenever possible.

8. Drill should be given in the analytic expression of functions stated in words. This leads to stating of equations.

9. The study of functions and equations should be continued as far as possible without special study of algebraic operations.

10. A critical discussion of the properties and the significance of the graph should be made.

11. The terms "function" and "variable" should be used persistently throughout the course.

12. Practical problems should be analysed for functional relations.

13. Special emphasis should be laid upon functional relations occurring in various parts of the subject-matter.

14. Trigonometric functions should be introduced in connection with similar triangles in geometry.

15. In an advanced course in algebra, functions should be classified and their general properties discussed.

In answer to the first objection mentioned above, we cite the fact that even the most radical promoters of the function movement do not propose to go into an elaborate "function theory" in the high school. It is proposed to present merely the simple, elementary notion of function and its applications.

That the simple notion of function is within the comprehension of high school students has repeatedly been pointed out by writers on the teaching of mathematics. Quotations to this effect from the writings of J. W. Young and J. W. A. Young are given in Chapter IV. H. Schülke,[1] one of the reform leaders in Germany, points out the inconsistency of the argument that the function notion is too difficult. He calls attention to the fact that the pupil makes use of graphic representation very early in his study of geography when he locates points on the earth's surface by means of their latitude and longitude. It may also be pointed out that the formulae of arithmetic are all expressions of functional relations.

Inasmuch as our plan of treating the function notion consists in an emphasis upon what is already in the algebra, rather than in the introduction of new material, the additional time required will not be very great. Of course, in view of the importance of the function notion, that part of the subject-matter of algebra which embodies this concept should be given more time and emphasis. More time should be given to the study of the graph, to the evaluation of functions, to the stating of functions in analytic form, etc. However, since it is our aim to make algebra more concrete and prac-

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1. Zeitschrift, Vol. 37, p 166.

tical, we would make room for extra emphasis upon functions by omitting some of the more abstract manipulation of symbolism. Among the parts to be curtailed we would suggest (a) complex fractions, (b) square root, (c) simultaneous quadratics.[1]

It is of the utmost importance that the student of mathematics obtain a correct notion of functionality when the idea is first presented to him. This result is not attained through the presentation in the high school texts in common use. These texts invariably present the concept through a definition, and a narrow, restricted one at that. We shall quote a few of these definitions[2] and point out their defects. "An expression involving one or several letters is called a function of these letters." - Schultze's Algebra. "An algebraic expression that depends upon another for its value is called a function of the latter." - Wentworth and Smith's Algebra. "An algebraic expression involving one or more letters is a function of the letters involved." - Hawkes, Louby and Touton, Algebra. All three of these definitions connect the function concept with an "expression," and two of them specify an algebraic expression. This is unnecessarily and unfortunately restrictive. The function notion is much broader than the notion of "expression." From our considerations in Chapter II. it is evident that the functional relations that enter most frequently into our daily activities are seldom such as can be expressed algebraically. To anybody to whom the function notion is presented in such a manner it will mean very little more than the traditional "expression." Hence nothing is gained by using a new name without acquiring a new idea. The fundamental properties of functions

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1. Hedrick, op. cit.
2. References to these texts are given in Chapter IV.

and their wide applications are entirely missed in such a presentation.

Another type of definition is the following from Milne's Algebra: "An expression that has different values corresponding to different values of x is called a function of x." This definition contains one idea that is not implied in the others, namely that of a change of the value of a function for a change of values of the variable. Even a mathematician like Professor E. J. Wilczynski uses this idea as a basis for the definition of a function in his text on trigonometry.[1] But such a definition is inadequate. For example, in the function cos x, it is possible to give x a doubly infinite number of values and still observe no change in the value of the function. The same is true of all periodic functions, and in a limited way to multiple-valued functions. While variation with change of the independent variable is a characteristic property of some functions, it can not be made the fundamental property of all functions.

"When one variable depends upon another for its value, we say the first is a function of the second," is the definition given in the Long-Brenke series. This definition is less objectionable than any of the others quoted, but it is not quite as broad as desirable. For instance, it is sometimes desirable to let the function be independent of any variable, or in other words, be a constant. Thus, if y is a linear function of x, the equation connecting the two represents a straight line. But to include the special cases of lines parallel to the axes, for example, y=5, it is convenient to be able to say that y is a function of x although it does not depend upon x for its value.

If a student is to appreciate the full meaning of the function notion and apply the concept in his thinking, he must not get the idea that

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1. Plane Trigonometry and Applications, Allyn and Bacon, 1914, p. 13

a function is something found in his algebra in connection with expressions in x and y. And this is precisely what most students understand by a function after having studied the ordinary text-book presentation. The notion of functionality should imply a correspondence between variable elements. It is this broad, general notion that the pupil can use in his daily activities. This is the notion of function expressed by the Dirichlet definition quoted in Chapter II. The School Algebra by Rietz and Crathorne gives precisely this definition. This is the only high school text which we have found using the Dirichlet definition. The method of deducing this definition will be discussed later in this chapter.

We shall now examine briefly some of the ways in which functions may be expressed. The most exact expression is, of course, the formula, as for example $i=pvt$, $\log x$, $S=\frac{1}{2} gt^2$. Although it may be permissible for the independent variables to assume any one of an infinite number of values, it is always possible, at least in the simpler cases, to calculate the value of the function for any of these values of the variable.

Another means of representing functions is the graph. It is not possible in the plane to represent the relations between more than two variables, nor more than three in space. Furthermore, the graph can at best be constructed to represent accurately only a finite number of relations between variables. Manifestly, if the variables may assume a very large or an infinite number of values, it is not possible to represent these all exactly by graph. It is therefore less ideal than the formula method considered above, inasmuch as it is limited in scope and accuracy. But for many purposes the graph is superior to the formula, for it is more vivid and exhibits such properties of the function as slope, maxima, and minima, which

are not so evident from the formula.

Corresponding sets of tables is another means of representing functions, although inferior to either of the above methods. Tables share with the graph the disadvantage of not representing the function for all possible values. They are not so well adapted to interpolation, are less vivid, and do not exhibit the other properties of the function mentioned in the discussion of the graph.

Still another method is a mere statement in words. Illustrations of this kind were given in Chapter II. Functional relations may be such that this is the only practical means of expressing them. Consider, for example, the following: Any number may be represented by a decimal with an infinite number of digits. For example, $\frac{1}{7}$ = .14275714.....; or 215 = 214.99..... If the digits of a number so written be interchanged by twos, beginning with the left hand digit, then a new number results, .41727541..... in the first case, and 129.499..... in the second example. Clearly, the numbers so obtained are functions of the original numbers.[1] It would be difficult, however, if not impossible, to express such a function in any other way than in words.

The simple notion of function may be introduced at the very beginning of the work in algebra. This can best be done through graphical representation of a variety of concrete examples. These should be drawn from within the range of the average pupil's immediate experience. Only such functions should be chosen as can readily be represented graphically. A simple one is the temperature-time relation for a period of, say, twenty-

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1. This illustration is given in H. Mangaldt, "Einführung in die Höhere Mathematik," Erster Band, page 287

four hours on July 4 of last year. Such a curve may be sketched on the board and presented for discussion as a whole, as representing the rise and fall of temperature, without any reference being made to the method of construction, axes, or co-ordinates except a zero-line. Questions will draw out the meanings of the maxima, and minima of the curve, the approximate time of day when the temperature was highest and lowest, when it changed from rising to falling, and conversely. The question might be asked, what change would have to be made in the curve to show that there was a sudden rainstorm of about half-hour's duration about one o'clock P. M., followed by a clear sky? Attention may be called to the fact that the United States Weather Bureau has instruments which automatically record the temperature in just such fashion and its rise and fall is then studied from the picture.

Following a similar line of discussion, other data may be graphically represented. After considering a number of familiar functions in this manner, the pupils may be told that these relations are called functions. In the cases considered, a certain quantity, in each example, is a function of the time.

After discussing several examples like the above, volunteers may be called for to represent the height of a shrub, or length of a vine as a function of the time. They may be told that the plant had a given height in January, did not grow till May, was checked by frost in October, and grew no more for the rest of the year. It may be added that it would be convenient to indicate along a base-line the months to which these heights correspond. Another volunteer may be called upon to represent by a curve the weight of a man during the period from birth to seventy-five years, showing the years by tens at the bottom. Undulations, if any, will be

accounted for by such factors as age or illness. The question may follow, how the curve would have to be drawn to exhibit the weight of a man who, in his fortieth year, met with an accident necessitating the amputation of a limb, followed by a long illness from which, however, he recovered completely.

From the consideration of examples like those mentioned above, the conclusion may be drawn that the graph is a language, similar to our spoken and written language. And the graph tells us a great many facts in a much shorter way than words can tell us. For instance, a verbal statement of facts, giving a boy's weight at birth, gains in early years, time of illness, and accidents later in life, does not give us as clear a picture of this man's weight at the various stages of his life as the curve just drawn does.

After becoming familiar with the graphical representation of some of his familiar experiences in the manner indicated above, other examples, such as the distance traversed by a train at uniform speed, may be sketched. In examples like this it can be shown that an accurate curve may be obtained by representing distances by means of perpendiculars to the base line at points corresponding to the time and drawing a curve through the upper ends of these perpendiculars. From this consideration the method of locating points on a graph without actually drawing these perpendiculars is but a next step. In the above example, the formula for the distance as a function of the time, $d=st$ (where s represents the speed) can be deduced. In this example, d is a function of t, or distance is a function of time. The graph and the formula are two ways of representing the same function.

After discussing a number of simple functions in the manner indicated above, the notion of a variable can be deduced from a more detailed

consideration of some of these functions. For example, in the first illustration cited, the pupil will readily understand that if t should represent the time (say, by hours), then t might have any one of twenty-five different values. Such symbols are called variables. Similarly, T may represent the temperature at the hour intervals. Pupils may be called upon to suggest other variables in the examples considered.

After the idea of variable is fixed, the fundamental notion of the function, namely, correspondence, should then be established from the cases considered. In each case it is seen that "when the value of the one variable t is given, a corresponding value of the other variable is determined." This should then be agreed upon as the formal statement of the meaning of a function. The formula d=st can then be tested by the above definition. Other illustrations of functions will then be freely contributed by different students. From this time on, the correspondence between variables should be distinctly emphasized in every function met with until the idea is firmly fixed.

In order to instil the habit of functional thinking, the pupil should be required to plot a considerable amount of familiar data. Even if the relations involved in such data can not be expressed analytically, they help fix the function notion and foster the habit of functional thinking. Proficiency in the construction and interpretation of the graph is of great value in representing and interpreting many of the student's common experiences as well as problems of every-day life. In his "Teaching of High School Mathematics," G. W. Evans comments upon the graph in the following manner: "It has become, in the course of the last generation, a mode of public expression; it is addressed to the intelligent; it should be a part,

therefore, of the education of the youth; and its place is in the mathematics course."[1]

The following list may prove suggestive in choosing problems to be represented graphically: Attendance-time of local high school; games won-time, by local athletic teams; graduation - time, of local high school; average grade-time, of individual students; hunger-time, of each individual student. Graphic work holds the interest of the boys and girls, especially if they can represent data taken from their own experience. The same holds for data which is of current public interest, as immigration-time, or age-death statistics. Such a treatment of algebra would stimulate interest, relate the work to real life, and train the mind in the habit of functional thinking.

It will be remembered that the treatment of functions outlined above is to be given at the beginning of the first course in algebra. In other words, algebra is to be begun with a study of functions graphically represented. We also cited an example in which the function involved could be expressed by a formula. The student will readily see that the formulae which he used in his study of arithmetic also represent functions. He should analyse them and point out the correspondence. Formulae involving only two variables, such as the areas of a circle, triangle, or rectangle can be plotted.

Following the discussion of functions represented by familiar formulae, drill in the stating of functions by means of symbols should be given. For example, John is three times as old as James. Represent James' age. Represent Albert's age if he is five years older than James. Or, if

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1. Page 41 f.

a cistern can be filled in x minutes, what part can be filled in one minute? In Collin's algebra two lists of such problems are given.[1] Such functions should also be represented graphically.

The consideration of such functions leads to the equation. In the case of functions of one variable, the equation results when the value of a function is given. The statement that a function equals a given number or function is called an equation. The value of the variable in the cases considered can then be found. Equations in two variables represent one variable as a function of the other. The graphic representation will clearly illustrate the correspondence between the variables.

By a proper choice of material the treatment of equations can be made quite extensive without any formal discussion of operations with symbols, negative numbers, or other algebraic concepts other than those used in arithmetic. Bradbury and Emery's[2] "Algebra for Beginners" is written on this plan. The first forty pages are devoted to equations before the traditional routine, usually presented first, is taken up. The treatment includes the solution of linear equations, integral and fractional, in one and two variables.

In his doctor's dissertation "Mathematics for Freshmen Students of Engineering," submitted to the Graduate School of the University of Chicago, Theodore Lindquist relates an experience in teaching high school algebra by centering all the work around the equation. He sums up[3] the results obtained from the use of this method as follows: "It created great interest in the work and produced, at the end of the year, a class capable of using

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1. Op. cit., pages 121, 136
2. Published by Thompson, Brown and Company, Chicago, 1894
3. Page 99

algebraic equations and of interpreting results obtained from their solution in a very satisfactory manner." These illustrations would indicate that a treatment of algebra founded upon functionality and centered around the equation would be entirely possible. New notions and operations may be introduced as they are needed.

At first the graph and its properties are best discussed more or less informally from general impressions. More detailed steps in the interpreting of the graph can gradually be introduced. All the information obtainable should be extracted. Frequently the only fact observed is the value of the function for a given value of the variable and other information is lost sight of.

Among the properties of the function represented by a graph, the following should be emphasized:[1] (a) When it has a given value; (b) when it is greater or less than a given value; (c) the ranges of values included during the period considered; (e) when it changes from increasing to decreasing, and vice versa, when it is growing more and more rapidly, and when more slowly.

Free use should be made of the terms "function" and "variable" instead of "expression" and "unknown" or "letter." Students should be directed, for example, to express the area of a circle as a function of the radius, rather than in terms of the radius. We do not claim that the use of the word function will, in itself, give the student an understanding of the idea back of it. However, we do claim that the use of a word in its proper place aids in gaining familiarity with the idea expressed by it. This is especially true if the idea is frequently analyzed and applied, as should be done in the

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1. Carson, Various Uses of Graphs, Mathematical Gazette, March 1914, page 267

case of functions. The student is more likely to associate the properties of a function with an algebraic expression if it is called a function than if it is not.

In the solution of practical problems the student should be encouraged to ask himself the question, "What are the factors that enter into the problem," followed by "Of what is each factor a function?" or "On what does it depend?" It has been said that one of the chief values of algebra lies in the fact that it furnishes a means of expressing and interpreting relations between variable quantities.[1] Considering natural laws as expressions of functional relations between variables, the method of solving problems involving natural laws can be generalized into the finding of the variables upon which the functions depend and stating their relations to each other.

In Chapter III. we quoted a number of references to the function relation in Schwab's "Geometrie." Such explicit mention of the functional relation should also be made whenever it is met with in the work throughout the algebra course. As instances in which functions should be emphasized we mention the following:

(1) Any algebraic expression is a function of the letters involved.

(2) A sum or product is a function of the addends and factors respectively.

(3) In any equation involving two or more variables, each variable is a function of the others.

(4) The roots of an equation are functions of the coefficients.

(5) In a trinomial square, the middle term is a function of the terms in the binomial that is squared.

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1. Hedrick, op cit.

(6) In any binomial expansion, the coefficient and the exponents of a given term are functions of the number of terms and the number of the term.

(7) In the product $(x+a)(x+b)$, the middle term and the last term are functions of a and b.

(8) For any point on a graph, the ordinate is a function of the abscissa.

In the treatment of similar triangles in geometry the trigonometric functions may easily be introduced and applied in the solution of the triangles. These functions are introduced and applied in this way by Stone-Millis,[1] but are called ratios and the functional relation between them and the angles is not mentioned.

In an advanced course in algebra, a chapter may well be devoted to a general discussion of functions. The various means of representing functions may here be discussed and the advantages and disadvantages of each noted. Students may be asked to suggest functions that lend themselves best to one or the other of these modes of expression, or suggest purposes for which some one way of representing a function would be more advantageous than the others. Algebraic functions should be classified on the basis of their degree. They may be told that there is another class of functions and simple illustrations of the logarithmetic and trigonometric functions cited. Such a treatment at this stage will serve to clarify and broaden the student's notion of function, suggest wider fields of application, and thus increase the usefulness of algebra as a means of interpreting his environment.

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1. Elementary Plane Geometry, p. 93, Benjamin H. Sanborn and Co. 1910

BIBLIOGRAPHY

Books:

Bradbury and Emery	Algebra for Beginners
Behrendsen-Götting	Lehrbuch der Mathematik
Carson	Mathematical Education
Collins	Practical Algebra
Duff	College Physics
Encyclopedie des Sciences Mathematiques, Vol. 2, Part 1	
Evans	Teaching of High School Mathematics
Hawkes, Louby and Touton,	First and Second Course in Algebra
Hedrick	Algebra for Secondary Schools
Höfler	Didaktik des Mathematischen Unterrichts
Kantor	Geschichte der Mathematik, Vol. III.
Klein und Schimmack	Der Mathematische Unterricht an den Höheren Schulen
Lindquist,	Mathematics for Freshmen Students of Engineering
Long-Brenke	Algebra, First Course
Mangoldt	Einführung in die Höhere Mathematik
Marshall	Economics and Industry
Milne	High School Algebra
Rietz and Crathorne	School Algebra
Stone-Millis	Elementary Plane Geometry
Schlichter	Elementary Mathematical Analysis

Schultze	Teaching of Mathematics in Elementary Schools
Schwab	Geometrie für die Mittleren Klassen der Realanstalten
Seligman	Principles of Economics
Slaught and Lennes	First Principles of Algebra
Smith	Teaching of Elementary Mathematics
Thompson and Bogart	An Exercise Book in the Economic History of the United States
Townsend and Goodenough	First Course in Calculus
Wentworth and Smith	School Algebra
Whitehead	Introduction to Mathematics
Wilczysiski	Plane Trigonometry
Young, J. W.	Fundamental Concepts of Mathematics
Young, J. W. A.	Teaching of Mathematics in Prussia
	Teaching of Mathematics
	Monographs on Modern Mathematics

Periodicals and Bulletins:

American Mathematical Monthly, March and April 1915

Bulletin American Mathematical Society, Vol. 19

Bulletin, U. S. Bureau of Education, No. 29 and No. 7

L'Enseignment Mathematique, Vol. 11

German International Committee Report, Band III., Heft 1

Illinois Agricultural College Experiment Station, Bulletins 26, 99, 146

Kansas " " " " Bulletin 171

Michigan " " " " Bulletin 5

Mathematical Gazette, Dec. 1911, Jan. 1912, Dec. 1912, Jan. 1913, Mar. 1914

School Review, Oct. 1914

School Science and Mathematics, Vol. 11

Science, Vol. 17

Thirteenth Yearbook, National Society for the Study of Education

University of Illinois Annual Register 1913-14 and 1914-15

Zeitschrift des Mathematischen und Naturwissenschaftlichen Unterrichts, Vol. 18, 38, 39, 41

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